Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty — I

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A complete set of bandlimited functions is described which possesses the curious property of being orthogonal over a given finite interval as well as over $(-\infty, \infty)$. *Properties of the functions are derived and several applications to the representation of signals are made.*

I. INTRODUCTION

It is pointed out in this paper that the eigenfunctions of the finite Fourier transform are certain prolate spheroidal wave functions. These eigenfunctions properly extended possess properties that make them ideally suited for the study of certain questions regarding the relationship between functions and their Fourier transforms. Here we shall study the functions in some detail and present some applications to the representation of bandlimited functions. The property that we shall be most concerned with is the orthogonality of the functions over two different intervals. The paper[1] by Landau and Pollak which follows draws on this material, establishes other properties of the functions and provides further examples of their application.

After some definitions contained in the next section, we proceed to state without proof in Section III our main results. Certain applications of these results are then given in Section IV. The remaining sections of the paper are devoted to establishing the results already stated.

II. NOTATION

In what follows, we denote by $\mathfrak{L}_\infty^2$ the class of all complex valued functions $f(t)$ defined on the real line and integrable in absolute square. We adopt the notation

$$\| f(t) \|_A^2 = \int_{-A}^{A} |f(t)|^2 \, dt \tag{1}$$

and refer to $\| f(t) \|_\infty^2$ as *the total energy of* $f(t)$ and refer to $\| f(t) \|_A^2$ as *the energy of* $f(t)$ *in the interval* $(-A,A)$. In an analogous manner, we denote by $\mathfrak{L}_A^2$ the class of all complex valued functions $f(t)$ defined for $-A \leqq t \leqq A$ and integrable in absolute square in the interval $(-A,A)$.

Functions in $\mathfrak{L}_\infty^2$ possess Fourier transforms. Upper and lower case versions of a letter will always denote a Fourier pair. We write, for example,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega, \tag{2}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt. \tag{3}$$

We refer to t as *time*, ω as *angular frequency* and $\omega/2\pi$ as *frequency*. The functions $F(\omega)$ are also integrable in absolute square. In this notation Parseval's theorem is

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)} \, d\omega. \tag{4}$$

We denote by $\mathfrak{B}$ the subclass of $\mathfrak{L}_\infty^2$ consisting of those functions, $f(t)$, whose Fourier transforms, $F(\omega)$, vanish if $| \omega | > \Omega$. Here $\Omega = 2\pi W$ is a positive real number fixed throughout this paper. Every member, $f(t)$, of $\mathfrak{B}$ can be written as a finite Fourier transform of a function integrable in absolute square:

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) e^{i\omega t} \, d\omega. \tag{5}$$

Functions in $\mathfrak{B}$ are called *bandlimited* and $\mathfrak{B}$ will be referred to as *the class of bandlimited functions*. It follows from (5) that members of $\mathfrak{B}$ are entire functions of the complex variable t.

From any function $f(t)$ in $\mathfrak{L}_\infty^2$ we can obtain a function, $Bf(t)$, contained in $\mathfrak{B}$ by the rule

$$Bf(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) e^{i\omega t} \, d\omega, \tag{6}$$

where $F(\omega)$ is given by (3). We call $Bf(t)$ the *bandlimited version* of $f(t)$. We regard B as an operator whose effect on a function in $\mathfrak{L}_\infty^2$ is to produce its bandlimited version. In electrical engineering terms, $Bf(t)$ results from passing $f(t)$ through an ideal low-pass filter with angular cutoff frequency Ω.

We denote by $\mathfrak{D}$ the subclass of functions, $f(t)$, of $\mathfrak{L}_\infty^2$ each of which vanishes for $| t | > T/2$. Here T is a positive real number fixed through-

out this paper. Members of $\mathfrak{D}$ are called *timelimited* and $\mathfrak{D}$ will be referred to as *the class of timelimited functions.*

From any function $f(t)$ in $\mathfrak{L}_\infty^2$ we can obtain a function $Df(t)$ contained in $\mathfrak{D}$ by the rule

$$Df(t) = \begin{cases} f(t), & |t| \leqq T/2 \\ 0, & |t| > T/2. \end{cases} \tag{7}$$

We call $Df(t)$ the *timelimited version* of $f(t)$. We regard D as an operator whose effect on a function of $\mathfrak{L}_\infty^2$ is to produce its timelimited version.

We shall use the notation $f(t) \in \mathfrak{F}$ to mean that the function $f(t)$ belongs to the class $\mathfrak{F}$ of functions.

III. RESULTS

The statements made below are proved in Sections V and VI.

Given any $T > 0$ and any $\Omega > 0$, we can find a countably infinite set of real functions $\psi_0(t),\psi_1(t),\psi_2(t),\cdots$ and a set of real positive numbers

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots \tag{8}$$

with the following properties:

i. The $\psi_i(t)$ are bandlimited, orthonormal on the real line and complete in $\mathfrak{B}$:

$$\int_{-\infty}^{\infty} \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \qquad i, j = 0, 1, 2, \cdots . \tag{9}$$

ii. In the interval $-T/2 \leqq t \leqq T/2$, the $\psi_i(t)$ are orthogonal and complete in $\mathfrak{L}_{T/2}^2$:

$$\int_{-T/2}^{T/2} \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0, & i \neq j \\ \lambda_i, & i = j \end{cases} \qquad i, j = 0, 1, 2, \cdots . \tag{10}$$

iii. For all values of t, real or complex,

$$\lambda_i\psi_i(t) = \int_{-T/2}^{T/2} \frac{\sin \Omega(t-s)}{\pi(t-s)}\,\psi_i(s)\,ds, \qquad i = 0, 1, 2, \cdots . \tag{11}$$

Further properties of the ψ's are given in Sections V and VI.

The notation used above conceals the fact that both the ψ's and the λ's are functions of the product ΩT. When it is necessary to make this dependence explicit, we write $\lambda_i = \lambda_i(c)$, $\psi_i(t) = \psi_i(c,t)$, $i = 0,1,2,\cdots$, where $2c = \Omega T$.

Some values of $\lambda_i(c)$ are given in Table I. It is to be noted that for a fixed value of c the λ_i fall off to zero rapidly with increasing i once i has

TABLE I—VALUES OF $\lambda_n(c) = L_n(c) \times 10^{-p_n(c)}$

n	$c = 0.5$		$c = 1.0$		$c = 2.0$		$c = 4.0$		$c = 8.0$	
	L	p	L	p	L	p	L	p	L	p
0	3.0969	1	5.7258	1	8.8056	1	9.9589	1	1.0000	0
1	8.5811	3	6.2791	2	3.5564	1	9.1211	1	9.9988	1
2	3.9175	5	1.2375	3	3.5868	2	5.1905	1	9.9700	1
3	7.2114	8	9.2010	6	1.1522	3	1.1021	1	9.6055	1
4	7.2714	11	3.7179	8	1.8882	5	8.8279	3	7.4790	1
5	4.6378	14	9.4914	11	1.9359	7	3.8129	4	3.2028	1
6	2.0413	17	1.6716	13	1.3661	9	1.0951	5	6.0784	2
7	6.5766	21	2.1544	16	7.0489	12	2.2786	7	6.1263	3
8	1.6183	24	2.1207	19	2.7768	14	3.6066	9	4.1825	4

exceeded $(2/\pi)c$. (The significance of this will be discussed in detail in a later paper.) Because of (9) and (10), namely $\| \psi_i \|_\infty^2 = 1$, $\| \psi_i \|_{T/2}^2 = \lambda_i$, a small value of λ_i implies that $\psi_i(t)$ will have most of its energy outside the interval $(-T/2,T/2)$ whereas a value of λ_i near 1 implies that $\psi_i(t)$ will be concentrated largely in $(-T/2,T/2)$. This behavior of the ψ's can be clearly seen in Figs. 1 through 5. Figs. 1 through 4 show $\psi_0(c,t)$, $\psi_1(c,t)$, $\psi_2(c,t)$ and $\psi_3(c,t)$ for several different values of c. For $c = 0.5$, or $(2/\pi)c = 0.3183$, as shown on Fig. 1, ψ_2 and ψ_3 are practically zero in the interval $(-T/2,T/2)$. For $c = 4$, or $(2/\pi)c = 2.546$, as shown on Fig. 4, ψ_0 is largely concentrated in the interval $(-T/2, T/2)$. Fig. 5 compares $\psi_0(c,t)$ for several different values of c.

IV. SOME APPLICATIONS

4.1 *Extrapolation of a Bandlimited Function*

It is sometimes desired to extrapolate a bandlimited function known only on the interval $(-T/2,T/2)$ to values outside this interval. Since any $f \in \mathfrak{B}$ is an entire function, this extrapolation can be done exactly in principle. One could, for example, calculate successive derivatives of f at some point in $(-T/2,T/2)$ and form a Taylor series representation which would converge everywhere. In practice, however, such a Taylor series would necessarily be truncated and the resultant approximation to $f(t)$ would be a polynomial which for sufficiently large values of $|t|$ would give a very poor approximation to f. This approximation is not, of course, bandlimited.

The functions ψ_i provide an alternative approach. Since $f \in \mathfrak{B}$, we can write, from i., for all t

$$f(t) = \sum_0^\infty a_n \psi_n(t), \tag{12}$$

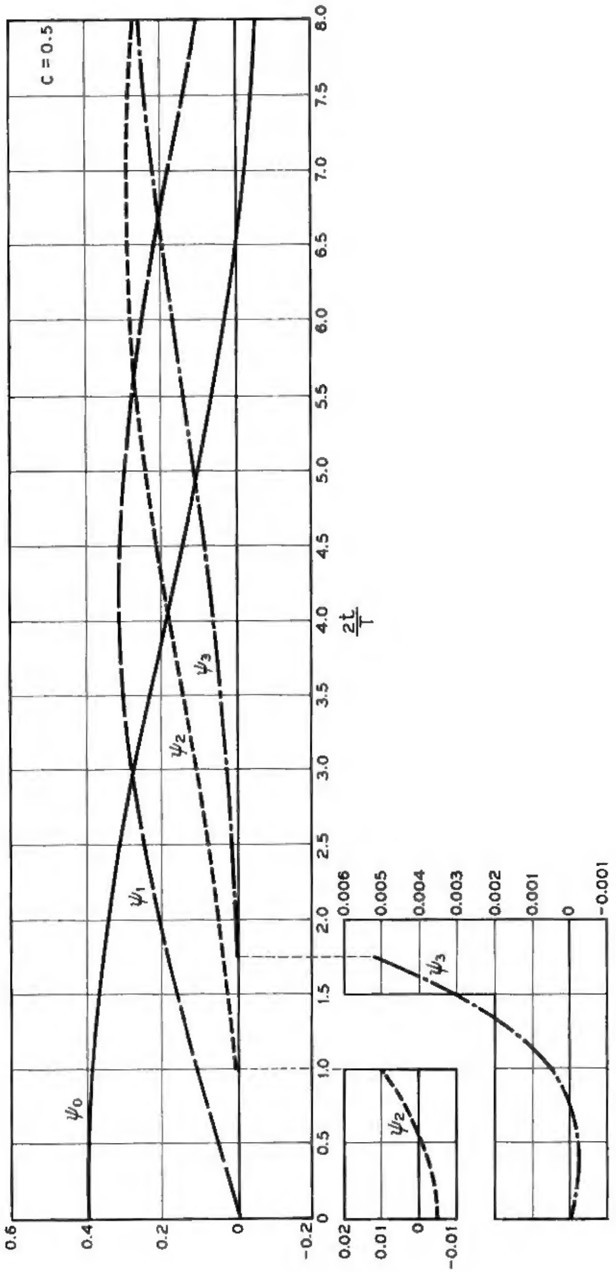

Fig. 1 — $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ vs. $2t/T$ for $c = 0.5$.

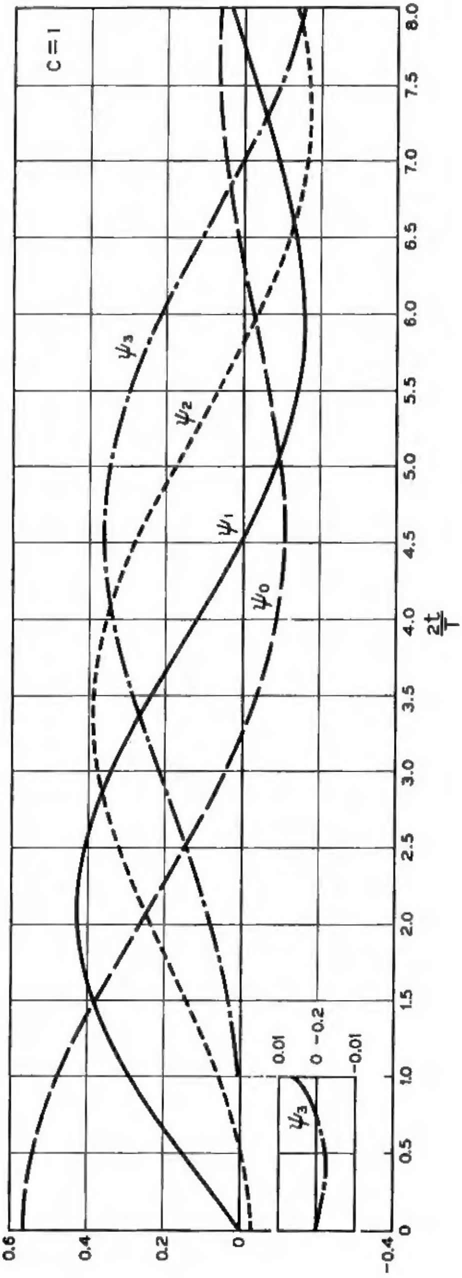

Fig. 2 — $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ vs. $2t/T$ for $c = 1.0$.

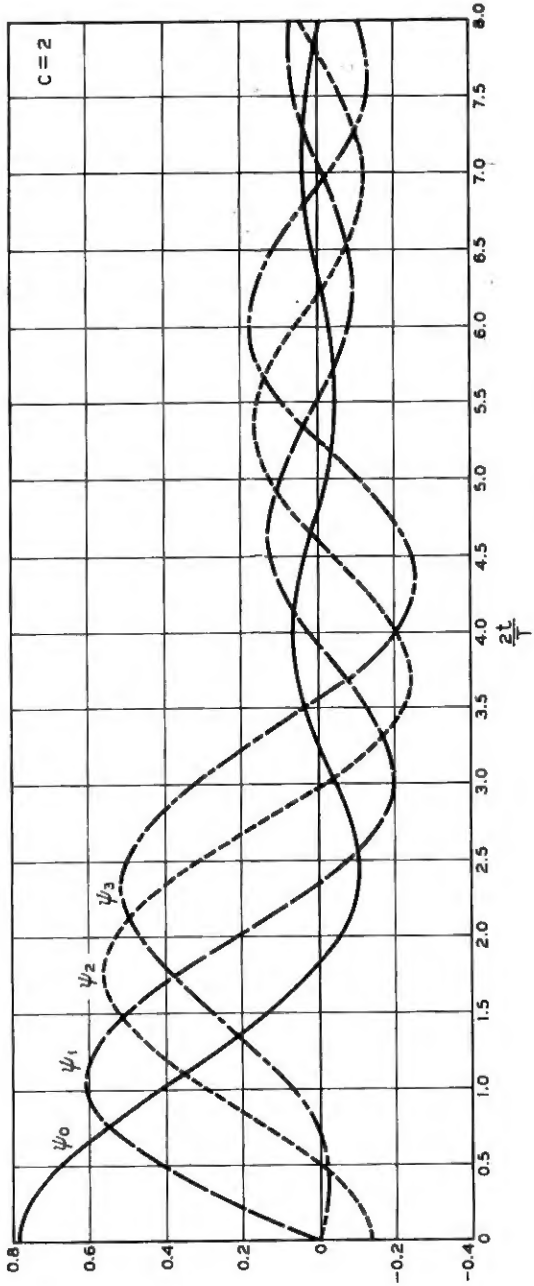

Fig. 3 — $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ vs. $2t/T$ for $c = 2.0$.

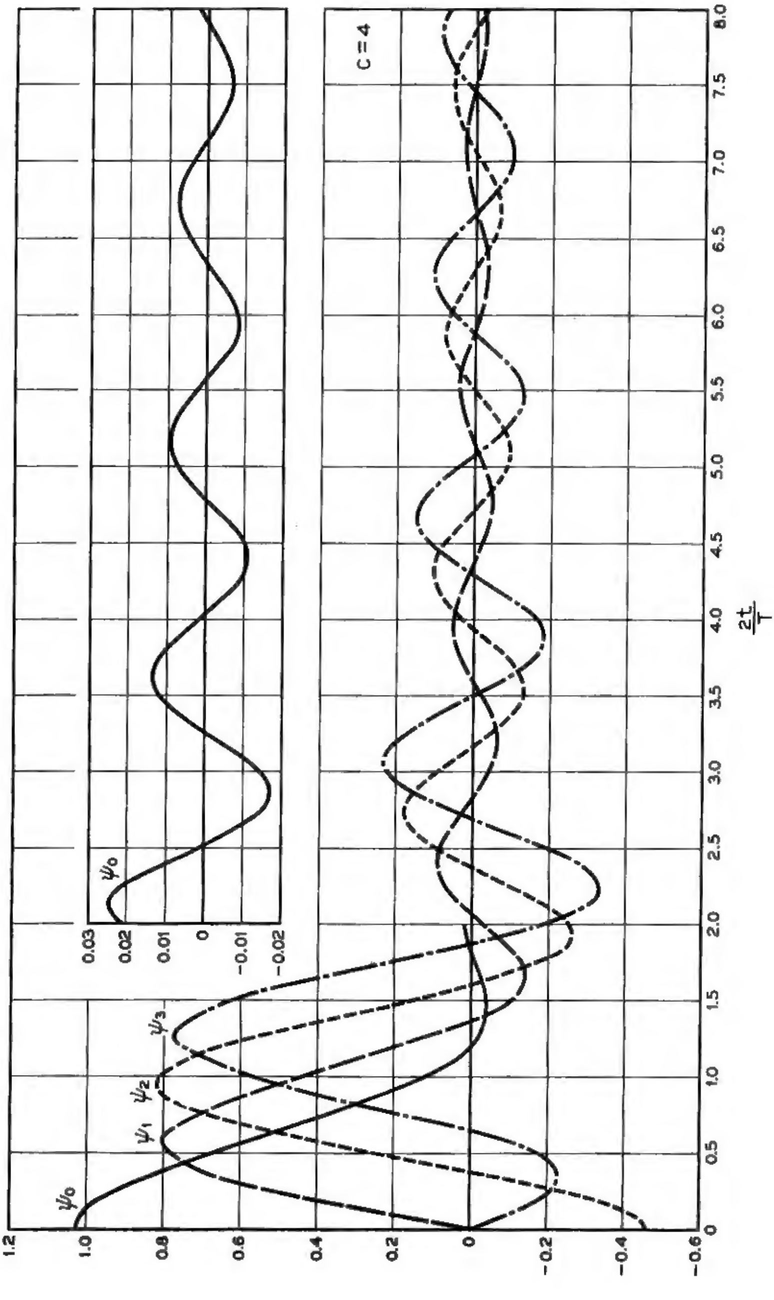

Fig. 4 — $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ vs. $2t/T$ for $c = 4.0$.

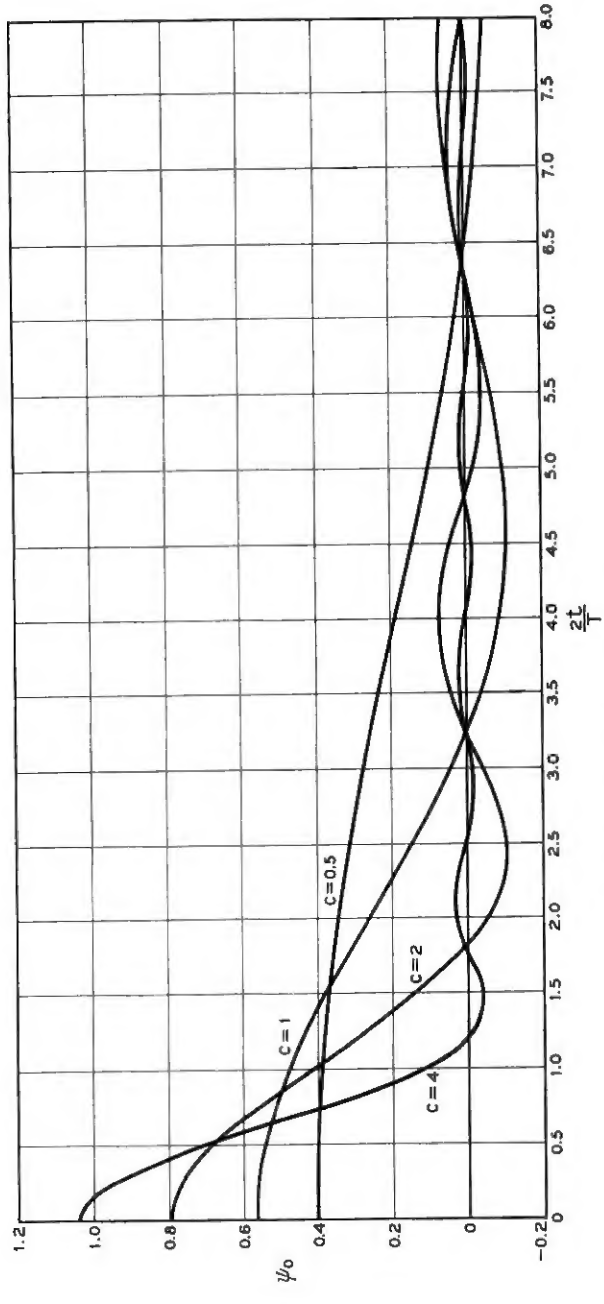

Fig. 5 — $\psi_0(c,t)$ vs. $2t/T$ for $c = 0.5, 1.0, 2.0, 4.0$.

where

$$a_n = \int_{-\infty}^{\infty} f(t)\psi_n(t)\,dt,$$
$$\sum_0^{\infty} a_n^2 = \int_{-\infty}^{\infty} f(t)^2\,dt \tag{13}$$

and the convergence in (12) is in the mean square sense

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left[f(t) - \sum^{N} a_n\psi_n(t) \right]^2 dt = 0.$$

Multiply (12) by $\psi_j(t)$, integrate and use (10). There results

$$a_n = \frac{1}{\lambda_n} \int_{-T/2}^{T/2} f(t)\psi_n(t)\,dt. \tag{14}$$

The coefficients in (12) *can be determined by* (14) *from values of* $f(t)$ *in the interval* $(-T/2,T/2)$.

The above result suggests approximating $f(t)$ for all t by

$$f_N(t) = \sum_0^{N} a_n\psi_n(t) \tag{15}$$

with the a_n given by (14). The approximation (15) is itself bandlimited. The mean squared error is

$$\int_{-\infty}^{\infty} [f(t) - f_N(t)]^2\,dt = \sum_{N+1}^{\infty} a_n^2 \tag{16}$$

and by (13) can be made as small as desired by making N sufficiently large. In the sense of (16), the extrapolation remains good for all t. The error in the fit of f_N to f in $(-T/2,T/2)$ is given by

$$\int_{-T/2}^{T/2} (f - f_N)^2\,dt = \sum_{N+1}^{\infty} a_n^2\lambda_n\,. \tag{17}$$

As the λ_n approach zero rapidly for sufficiently large n, it may happen that (17) is small for values of N for which (16) is still large. The fit of f_N inside the interval should not be taken as an indication of the fit elsewhere.

4.2 *Approximation in an Interval by a Bandlimited Function*

Suppose now $f(t) \in \mathfrak{L}_{T/2}^2$ is known in the interval $(-T/2,T/2)$ but f is not necessarily a piece of a bandlimited unction. From i. above it

follows that $f(t)$ may still be represented by (12) with a's given by (14), but this representation is valid now only for $|t| \leqq T/2$. If indeed f is not a piece of a bandlimited function, the series (12) will certainly not converge in mean square over the whole real line.

The foregoing suggests the utility of finite sums of the form (15) as approximants to bandlimited functions having a prescribed form in the interval $(-T/2,T/2)$. The conditions of bandlimitation and prescribed form in $(-T/2,T/2)$ are, of course, in general incompatible (unless indeed, the prescribed form is a piece of a bandlimited function). However, finite sums of the form (15) taken for all t with a's computed by (14) permit approximations by bandlimited functions to a prescribed $f \in \mathfrak{L}_{T/2}^2$. We are assured by ii. that the approximation can be made as good as desired in the sense that the right side of (17) approaches zero for large N. We have, however,

$$\int_{-\infty}^{\infty} f_N^2(t)\,dt = \sum_0^N a_n^2$$

and, if f is not a piece of a bandlimited function, $\sum^N a_n^2$ grows without bound for increasing N. Thus, in approximating a piece of a nonbandlimited function by a bandlimited function, we exchange goodness of fit in $(-T/2,T/2)$ with wildness of behavior outside this interval.

We now impose an energy restriction. Given $f \in \mathfrak{L}_{T/2}^2$. What $g \in \mathfrak{B}$ with prescribed energy $\| g \|_\infty^2 = E$ minimizes $\| f - g \|_{T/2}^2$? Let

$$f = \sum a_n\psi_n(t), \qquad |t| \leqq T/2,$$
$$g = \sum b_n\psi_n(t), \qquad -\infty < t < \infty.$$

Then a simple argument gives

$$b_n = \frac{a_n\lambda_n}{\mu + \lambda_n},$$

where μ is the unique positive number which satisfies

$$E = \sum \frac{a_n^2\lambda_n^2}{(\mu + \lambda_n)^2}.$$

If the constraint on g is that the energy outside $(-T/2,T/2)$ is prescribed, $\| g \|_\infty^2 - \| g \|_{T/2}^2 = E'$, rather than the total energy, the result is

$$b_n = \frac{a_n\lambda_n}{\mu(1 - \lambda_n) + \lambda_n},$$

where μ (again positive) is chosen to satisfy

$$E' = \sum \frac{a_n^2 \lambda_n^2}{[\mu(1 - \lambda_n) + \lambda_n]^2}.$$

4.3 *Some Extremal Properties of $\psi_0(t)$*

The ψ's possess a number of interesting extremal properties. The most important of these, the fact that ψ_0 has the largest energy in $(-T/2, T/2)$ of all function in $\mathcal{B}$ of unit total energy, is discussed in detail by Landau and Pollak.[1] We comment here on two other extremal properties of ψ_0.

Let $f(t) \in \mathcal{L}_\infty^2$ have total energy $E = \| f \|_\infty^2$. The timelimited version of $f(t)$ has total energy $E_D = \| Df \|_\infty^2 = \| f \|_{T/2}^2 \leqq E$. Since Df cannot be bandlimited, its Fourier transform has nonvanishing energy in $| \omega | > \Omega$. The bandlimited version of Df, namely BDf, will therefore have total energy $E_{BD} < E_D \leqq E$. The operation BD transforms a member of $\mathcal{L}_\infty^2$ into a member of $\mathcal{B}$ with smaller total energy. Which members of $\mathcal{L}_\infty^2$ lose the smallest fraction of their energy under such a transformation? That is, for which $f \in \mathcal{L}_\infty^2$ is $\mu \equiv \| BDf \|_\infty^2 / \| f \|_\infty^2$ a maximum?

The answer to this question, unique except for an arbitrary multiplicative constant, is $D\psi_0(t)$. This may be seen as follows. From (3), (6) and the definition (7) of D,

$$\begin{aligned} BDf(t) &= \frac{1}{2\pi} \int_{-\Omega}^{\Omega} d\omega \, e^{i\omega t} \int_{-T/2}^{T/2} ds \, f(s) \, e^{-i\omega s} \\ &= \int_{-T/2}^{T/2} \rho_\Omega(t - s) f(s) \, ds, \end{aligned} \tag{18}$$

where we have written

$$\rho_\Omega(\tau) = \frac{\sin \Omega\tau}{\pi\tau} = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} d\omega \, e^{i\omega\tau}. \tag{19}$$

Note that $\rho_\Omega(\tau)$ is an even function of τ and that from (19) and Parseval's theorem (4) it follows that

$$\int_{-\infty}^{\infty} \rho_\Omega(t - u) \rho_\Omega(u - s) \, du = \rho_\Omega(t - s). \tag{20}$$

Therefore,

$$\| BDf(u) \|_\infty^2 = \int_{-\infty}^{\infty} du\, [BDf(u)]\overline{[BDf(u)]}$$

$$= \int_{-\infty}^{\infty} du \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds\, \rho_\Omega(t-u)\rho_\Omega(u-s) f(t)\bar{f}(s) \quad (21)$$

$$= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds\, \rho_\Omega(t-s) f(t)\bar{f}(s).$$

Here we have used (20) and the fact that ρ_Ω is real and even.

Since from (21) we see that $\| BDf \|_\infty^2$ depends only on values of f in $(-T/2,T/2)$, it follows that μ is equal to the maximum of

$$\nu = \frac{\| BDf \|_\infty^2}{\| f \|_{T/2}^2} = \frac{\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds\, \rho_\Omega(t-s) f(t)\bar{f}(s)}{\int_{-T/2}^{T/2} | f(t) |^2\, dt}$$

over all $f \in \mathfrak{L}_{T/2}^2$. It is well known that the solution to this problem is $\nu = \lambda_0$, where λ_0 is the largest eigenvalue of the integral equation

$$\lambda f(t) = \int_{-T/2}^{T/2} \rho_\Omega(t-s) f(s)\, ds, \qquad | t | \leqq T/2, \qquad (22)$$

and that ν attains the value λ_0 for f equal to a corresponding eigenfunction. We shall see later that ψ_0 is such an eigenfunction. Thus f agrees with ψ_0 in $(-T/2,T/2)$ and so $D\psi_0$ is a function in $\mathfrak{L}_\infty^2$ for which μ attains its maximum value λ_0.

We now ask which $f \in \mathfrak{B}$ as opposed to $f \in \mathfrak{L}_\infty^2$ maximizes μ. That is, which *bandlimited* function loses the least (fractional) energy when first timelimited then bandlimited? The answer is ψ_0 and the corresponding value of μ is λ_0^2.

To see this, introduce the representation (5) for $f \in \mathfrak{B}$ into the numerator [as given by (21)] of μ. There results

$$\| BDf \|_\infty^2 = \frac{1}{4\pi^2} \int_{-\Omega}^{\Omega} d\omega \int_{-\Omega}^{\Omega} d\omega'\, F(\omega)\overline{F(\omega')} K(\omega,\omega'),$$

where we have set

$$K(\omega,\omega') = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\, \rho_\Omega(t-t') e^{i\omega t}\, e^{-i\omega' t'}.$$

To transform this expression further, introduce the representation (19) to obtain

$$\begin{aligned} K(\omega,\omega') &= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} d\omega'' \int_{-T/2}^{T/2} dt\, e^{it(\omega-\omega'')} \int_{-T/2}^{T/2} dt'\, e^{-it'(\omega'-\omega'')} \\ &= 2\pi \int_{-\Omega}^{\Omega} d\omega'' \rho_{T/2}(\omega-\omega'')\rho_{T/2}(\omega''-\omega') \\ &= 2\pi\rho_{T/2}^{(2)}(\omega,\omega'). \end{aligned}$$

By Parseval's theorem, (4), the denominator of μ can be written as

$$\| f \|_{\infty}^{2} = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} |F(\omega)|^2\, d\omega.$$

Our task, then, is to maximize

$$\mu = \frac{\| BDf \|_{\infty}^{2}}{\| f \|_{\infty}^{2}} = \frac{\int_{-\Omega}^{\Omega} d\omega \int_{-\Omega}^{\Omega} d\omega' \rho_{T/2}^{(2)}(\omega,\omega')F(\omega)\bar{F}(\omega')}{\int_{-\Omega}^{\Omega} |F(\omega)|^2\, d\omega}$$

over all $F \in \mathfrak{L}_{\Omega}^{2}$. The solution to this problem is $\mu = \mu_0$, where μ_0 is the largest eigenvalue of the integral equation

$$\lambda F(\omega) = \int_{-\Omega}^{\Omega} \rho_{T/2}^{(2)}(\omega,\omega')F(\omega')\, d\omega'.$$

Now $\rho_{T/2}^{(2)}(\omega,\omega')$ is the first iterate of $\rho_{T/2}(\omega-\omega')$. Therefore, μ_0 is the square of the largest eigenvalue of the integral equation

$$\lambda F(\omega) = \int_{-\Omega}^{\Omega} \rho_{T/2}(\omega-\omega')F(\omega')\, d\omega'.$$

A change of variables reduces this equation to the form of (22) whence it is seen that $\mu = \lambda_0^2$ and that $F(\omega) = \psi_0(\omega T/2\Omega)$ for $|\omega| \leqq \Omega$. From (29), which will be established later, it follows that $f(t) = \psi_0(t)$.

4.4 *Problems Concerning Bandlimited Noise*

Much of the theory of detection, parameter estimation and prediction of signals in noise when observations are made in a finite time is based on the Karhunen-Loève representation of the noise. (See Ref. 2 for such a treatment of these problems.) This representation involves expansions in terms of the eigenfunction solutions of a certain integral equation. When the noise in question is second order stationary and with angular frequency spectral density uniform in $(-\Omega,\Omega)$ and zero elsewhere (bandlimited white noise), the integral equation in question is identical with (19), (22). The function ψ_i and eigenvalues λ_i thus play an important role in numerous questions concerning bandlimited white

noise observed for a finite time. Their role in this connection has been pointed out previously in Ref. 3.

V. THE PROLATE SPHEROIDAL WAVE FUNCTION

The functions $\psi_i(c,t)$ are scaled versions of certain of the angular prolate spheroidal wave functions. A number of books[4,5,6,7] treat the prolate spheroidal wave functions in detail. We will draw freely from this literature. We adopt the notation* of Flammer.[4]

When c is real, the differential equation

$$(1 - t^2)\frac{d^2u}{dt^2} - 2t\frac{du}{dt} + (\chi - c^2t^2)u = 0 \tag{23}$$

has continuous solutions in the closed t interval $[-1,1]$ only for certain discrete real positive values $0 < \chi_0(c) < \chi_1(c) < \chi_2(c) < \cdots$ of the parameter χ. Corresponding to each eigenvalue $\chi_n(c)$, $n = 0,1,2, \cdots$ there is a unique solution $S_{0n}(c,t)$ such that $S_{0n}(c,0) = P_n(0)$ where $P_n(t)$ is the nth Legendre polynomial. The functions $S_{0n}(c,t)$ are called *angular prolate spheroidal functions.* They are real for real t, are continuous functions of c for $c \geqq 0$, and can be extended to be entire functions of the complex variable t. They are orthogonal in $(-1,1)$ and are complete in $\mathfrak{L}_1^2$. $S_{0n}(c,t)$ has exactly n zeros in $(-1,1)$, reduces to $P_n(t)$ uniformly in $[-1,1]$ as $c \to 0$, and is even or odd according as n is even or odd, $n = 0,1,2, \cdots$. The eigenvalues $\chi_n(c)$ are continuous functions of c and $\chi_n(0) = n(n + 1)$, $n = 0,1,2, \cdots$.

A second set of solutions $R_{0n}^{(1)}(c,t)$, $n = 0, 1, \cdots$, called *radial prolate spheroidal functions,* which differ from the angular functions only by a real scale factor,

$$R_{0n}^{(1)}(c,t) = k_n(c)S_{0n}(c,t),$$

are of use in many applications. These radial functions are normalized so that

$$R_{0n}^{(1)}(c,t) \to \frac{1}{ct}\cos\left[ct - \tfrac{1}{2}(n + 1)\pi\right]$$

as $t \to \infty$.

The equations

$$\frac{2c}{\pi}[R_{0n}^{(1)}(c,1)]^2 S_{0n}(c,t) = \int_{-1}^{1} \frac{\sin c(t - s)}{\pi(t - s)} S_{0n}(c,s)\,ds, \tag{24}$$

$$2i^n R_{0n}^{(1)}(c,1) S_{0n}(c,t) = \int_{-1}^{1} e^{icts} S_{0n}(c,s)\,ds \qquad n = 0, 1, 2, \cdots \tag{25}$$

* The reader should be cautioned that various authors disagree not only on notation for these functions, but also in their method of normalization.

are both special cases of more general integral relations satisfied by prolate spheroidal functions that can be found in the literature. They are valid for all t, real or complex.

Equation (24) shows that $S_{0n}(c,t)$ is a solution of the integral equation

$$\lambda f(t) = \int_{-1}^{1} \rho_c(t-s) f(s)\, ds, \qquad |t| \leqq 1 \tag{26}$$

corresponding to the eigenvalue

$$\lambda_n(c) = \frac{2c}{\pi} [R_{0n}^{(1)}(c,1)]^2, \qquad n = 0, 1, 2, \cdots . \tag{27}$$

Here $\rho_c(\tau)$ is given by (19). Indeed, the completeness of the S_{0n} in $\mathfrak{L}_1^2$ assures us that the quantities (27) are the only eigenvalues of (26) and that if these quantities are distinct, the S_{0n} are (apart from multiplicative constants) the unique $\mathfrak{L}_1^2$ solutions of (26). If several of the quantities (27) are equal for different values of n, then linear combinations of the corresponding S_{0n} will also satisfy (26). Within the sense of this degeneracy, then, the S_{0n} are unique solutions of (26). In Section VI we shall see, indeed, that this degeneracy does not occur.

Equation (19) and Bochner's theorem (Ref. 8, Theorem 23, p. 95) show that the kernel of (26) is positive definite. The quantities (27) are therefore strictly positive. Set

$$[u_n(c)]^2 = \int_{-1}^{1} [S_{0n}(c,t)]^2\, dt.$$

We now finally define

$$\psi_n(c,t) = \frac{\sqrt{\lambda_n(c)}}{u_n(c)} S_{0n}(c,2t/T). \tag{28}$$

Properties ii. of Section III now follow directly from definitions and the orthonormality and completeness of the S_{0n} in $(-1,1)$.

A change of variables and the definitions (27) and (28) convert (24) into (11). A change of variables converts (25) into

$$\frac{i^n \Omega R_{0n}^{(1)}(c,1)}{\pi} \psi_n(c,t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{i\omega t} \psi_n(c,\omega T/2\Omega)\, d\omega, \tag{29}$$

which shows $\psi_n \in \mathfrak{B}$. Indeed, since the function $\psi_n(c,\omega T/2\Omega)$ are complete in $-\Omega \leqq \omega \leqq \Omega$, Parseval's theorem shows that the $\psi_n(t)$ are complete in $\mathfrak{B}$. The remaining assertion of i. of Section III, namely (9),

follows from a computation. From (11) we have

$$\int_{-\infty}^{\infty} dt\,\psi_i(t)\psi_j(t)$$

$$= \frac{1}{\lambda_i\lambda_j}\int_{-\infty}^{\infty} dt \int_{-T/2}^{T/2} ds\,\rho_\Omega(t-s)\psi_i(s)\int_{-T/2}^{T/2} du\,\rho_\Omega(t-u)\psi_j(u)$$

$$= \frac{1}{\lambda_i\lambda_j}\int_{-T/2}^{T/2} du \int_{-T/2}^{T/2} ds\,\psi_i(s)\psi_j(u)\int_{-\infty}^{\infty} dt\,\rho_\Omega(u-t)\rho_\Omega(t-s)$$

$$= \frac{1}{\lambda_i\lambda_j}\int_{-T/2}^{T/2} du\,\psi_j(u)\int_{-T/2}^{T/2} ds\,\rho_c(u-s)\psi_i(s)$$

$$= \frac{1}{\lambda_j}\int_{-T/2}^{T/2} du\,\psi_j(u)\psi_i(u) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Here we have used (20) and (10).

All properties of the ψ's asserted in Section III have now been established except for (8). To this end we devote the next section.*

VI. NONDEGENERACY AND ORDERING OF THE EIGENVALUES OF (26)

We have seen that the $S_{0n}(c,t)$ are solutions of (26) with eigenvalues given by (27). We show now that we cannot have two distinct S_{0n} belonging to the same eigenvalue λ if $c > 0$.

Let $f_1(t)$ and $f_2(t)$ be two linearly independent solutions of (26) for the same λ, $c \neq 0$. Then

$$\lambda f_1(t) = \int_{-1}^{1} \rho_c(t-s) f_1(s)\,ds, \tag{30}$$

$$\lambda f_1'(t) = \int_{-1}^{1} \rho_c'(t-s) f_1(s)\,ds, \tag{31}$$

$$\lambda f_1''(t) = \int_{-1}^{1} \rho_c''(t-s) f_1(s)\,ds, \tag{32}$$

$$\lambda f_2(t) = \int_{-1}^{1} \rho_c(t-s) f_2(s)\,ds \tag{33}$$

* Ville and Bouzitat[9] recognized (independently of the earlier Ref. 3) that the solutions of the integral equation (11) are prolate spheroidal functions. They assert that the eigenvalues λ_n are ordered as in (8) when ψ_n is identified with S_{0n} but no proof of this fact appears in their paper or apparently elsewhere in the literature.

and

$$\lambda f_2''(t) = \int_{-1}^{1} \rho_c''(t - s) f_2(s)\, ds. \tag{34}$$

Assume now that f_1 is even and f_2 is odd. Integrate (31) by parts to obtain

$$\lambda f_1'(t) = f_1(1)[\rho_c(-1 - t) - \rho_c(1 - t)] + \int_{-1}^{1} \rho_c(t - s) f_1'(s)\, ds.$$

Multiply this equation by $f_2(t)$ and integrate to obtain

$$\begin{aligned}\lambda \int_{-1}^{1} f_2(t) f_1'(t)\, dt = \lambda f_1(1)[f_2(-1) - f_2(1)]& \\ + \int_{-1}^{1} dt \int_{-1}^{1} ds\, \rho_c(t - s) f_1'(s) f_2(t).&\end{aligned} \tag{35}$$

Now multiply (33) by $f_1'(t)$, integrate and subtract the result from (35). One finds $\lambda f_1(t)[f_2(-1) - f_2(1)] = 0$, or

$$f_1(1) f_2(1) = 0, \qquad f_1 \text{ even}, \quad f_2 \text{ odd}. \tag{36}$$

Assume now that $f_1(t)$ and $f_2(t)$ are of the same parity, i.e., both even or both odd. Multiply (32) by $f_2(t)$, multiply (34) by $f_1(t)$, subtract and integrate. There results

$$\begin{aligned}\lambda \int_{-1}^{1} dt(f_1''f_2 - f_2''f_1) &= \lambda \int_{-1}^{1} dt \frac{d}{dt}(f_1'f_2 - f_2'f_1) \\ &= 2\lambda[f_1'(1)f_2(1) - f_2'(1)f_1(1)] = 0\end{aligned}$$

or

$$f_1(1) f_2'(1) = f_2(1) f_1'(1), \qquad f_1 \text{ and } f_2 \text{ of same parity}. \tag{37}$$

For any two linearly independent solutions of (26) belonging to the same eigenvalue we must have either (36) or (37) hold. But we shall show that both of these conditions are impossible for two different S functions, say $S_{0n}(c,t)$ and $S_{0m}(c,t)$. From the differential equation (23), we see that

$$2S_{0n}'(1) = (\chi_n - c^2) S_{0n}(1). \tag{38}$$

If $S_{0n}(1)$ vanishes, then so does $S_{0n}'(1)$. But differentiating (23) shows

that if $S_{0n}(1)$ and $S'_{0n}(1)$ vanish so does $S''_{0n}(1)$. Repeated differentiation (which is possible since the S_{0n} are entire) shows that if $S_{0n}(1) = 0$, then $S_{0n}(t) \equiv 0$. Therefore condition (36) cannot hold. On the other hand, since $S_{0n}(1) \neq 0$, $S_{0m}(1) \neq 0$, (37) can be written

$$\frac{S'_{0m}(1)}{S_{0m}(1)} = \frac{S'_{0n}(1)}{S_{0n}(1)}$$

or

$$\frac{\chi_m - c^2}{2} = \frac{\chi_n - c^2}{2} \tag{39}$$

from (38). However, it is known that the eigenvalues of the differential equation (23) are nondegenerate if c is real, so that (39) cannot hold if $m \neq n$. The eigenvalues (27) are thus seen to be distinct.

By their definition, the S_{0n} functions are indexed so that the eigenvalues of the differential equation (23) $\chi_0 < \chi_1 < \chi_2 < \cdots$ are monotone increasing functions of their index. We have defined ψ_n in terms of the S_{0n} by (28) and have labeled the corresponding eigenvalue of (26) λ_n by (27). There remains the task of proving that the λ_n are ordered as in (8).

Our argument makes use of the fact (just demonstrated) that for all real $c \neq 0$ the $\lambda_n(c)$ are nondegenerate and the fact (see for example Ref. 10, vol. I, p. 128) that the eigenfunctions and eigenvalues of (26) are continuous functions of its kernel. Thus if we can prove that for some $c > 0$,

$$\lambda_0(c) > \lambda_1(c) > \lambda_2(c) \cdots,$$

then continuity and nondegeneracy of the λ's allows us to assert this ordering for all positive c.

We now establish this ordering for c sufficiently near zero. Let ψ_n and ψ_{n+1} be successive eigenfunctions of (26), $c \neq 0$. Then

$$\lambda_n \psi'_n(t) = \int_{-1}^{1} \rho'_c(t - s)\psi_n(s)\, ds,$$

$$\lambda_{n+1} \psi'_{n+1}(t) = \int_{-1}^{1} \rho'_c(t - s)\psi_{n+1}(s)\, ds.$$

Multiply the first of these equations by $\lambda_{n+1}\psi_{n+1}(t)$, multiply the second by $\lambda_n\psi_n(t)$, add the results and integrate to obtain

$$\lambda_n\lambda_{n+1}\int_{-1}^{1}(\psi_n'\psi_{n+1}+\psi_{n+1}'\psi_n)\,dt$$

$$= \lambda_{n+1}\int_{-1}^{1}dt\int_{-1}^{1}ds\,\rho_c'(t-s)\psi_{n+1}(t)\psi_n(s)$$

$$+ \lambda_n\int_{-1}^{1}dt\int_{-1}^{1}ds\,\rho_c'(t-s)\psi_n(t)\psi_{n+1}(s)$$

$$= (\lambda_n-\lambda_{n+1})\int_{-1}^{1}dt\int_{-1}^{1}ds\,\rho_c'(t-s)\psi_n(t)\psi_{n+1}(s)$$

$$= (\lambda_n-\lambda_{n+1})\lambda_{n+1}\int_{-1}^{1}\psi_n(t)\psi_{n+1}'(t)\,dt$$

or

$$\lambda_n - \lambda_{n+1} = \lambda_n\left(1+\frac{\int_{-1}^{1}\psi_n'\psi_{n+1}\,dt}{\int_{-1}^{1}\psi_n\psi_{n+1}'\,dt}\right). \tag{40}$$

Now as $c \to 0$, $\psi_n \to P_n(t)$, the nth Legendre polynomial, and $\psi_n' \to P_n'(t)$. The denominator of the fraction in (40) approaches

$$\int_{-1}^{1}P_nP_{n+1}'\,dt = P_nP_{n+1}\Big|_{-1}^{1} - \int_{-1}^{1}P_{n+1}P_n'\,dt = 2$$

since the integral on the right vanishes and $P_n(1) = 1$. The numerator approaches

$$\int_{-1}^{1}P_n'P_{n+1}\,dt = 0.$$

By making c sufficiently small, therefore, the fraction on the right of (40) is of absolute value less than unity and $\lambda_n - \lambda_{n+1} = \lambda_n[1 + 0(1)] \geqq 0$. Since for $c \neq 0$ the λ_n are all distinct and positive, the ordering (8) must hold. The limiting eigenvalues for $c \to 0$ are $0 = \lambda_0 = \lambda_1 = \lambda_2 = \cdots$.

VII. COMMENTS

It is worth pointing out that the basic importance of the ψ_n for the study of the relation between functions and their Fourier transforms stems from (25), which shows that the S_{0n} are eigenfunctions of the finite Fourier transform kernel. Indeed, many of the important properties of the ψ's (i. and ii. of Section III, for example) follow directly from (25)

or its first iterate (24), without explicit use of (23) or recognition of the S_{0n} as angular prolate spheroidal wave functions.

In the interests of simplicity of presentation, we have not put forth the theme of this work in its most general form. We here make just one comment in this direction and leave other generalizations to the interested reader. The curious orthogonality over two different pointsets of the analytically continued solution of (22) will hold whenever (20) is true and the solutions are in $\mathcal{L}_\infty^2$. For example, if the kernel $\rho(\tau)$ of (22) is even and has a Fourier transform constant on intervals and zero elsewhere, e.g., $\rho_1(\tau) = \rho_0(\tau) \cos \alpha\tau$, $\alpha > \Omega$, then the double orthogonality maintains. The eigenfunctions for the bandpass kernel $\rho_1(\tau)$ do not seem to be expressible in terms of well-studied functions. Computations in this case indicate the existence of degenerate eigenvalues.

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